Regularity for a degenerate nonlocal equation

modeling the evolution of active particles

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SCUOLA NORMALE SUPERIORE

Anacapri Workshop: Nonlocal nonlinear PDE

joint work with Luca Alasio (Paris VI) and Martin Burger (Hamburg)

10^{th} July 2025

Contents

Contents

Where does the eq. come from?

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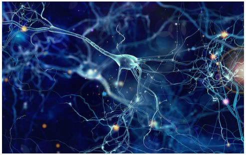

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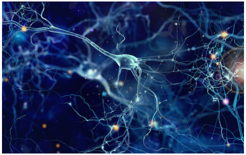

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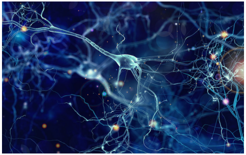

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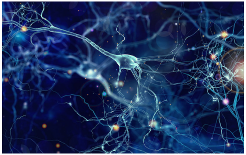

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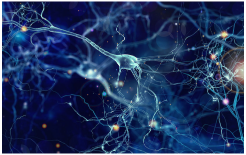

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All these can be modeled using interacting particle systems

(Google Images)

Active Brownian system

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- Focus of talk: **regularity** for this PDE

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(div and ∇ only wrt x)

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... also for $\psi \in C^\infty_c(\mathbb{R}^3)$ with integrals over all $\mathbb{R}^3$

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Let $T > 0$ be arbitrary. Let f_0 be non-negative initial data such that $f_0 \in L^p(\mathbb{T}^3)$ for some $p > 1$ and

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Theorem (Alasio, S. '25 *NoDEA*)

With "reasonable" structural assumptions on the initial data, any weak solution is smooth: $f \in C^\infty((0, T) \times \mathbb{T}^3)$.

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$$\rho[u] = \frac{\int_0^{2\pi} e^u \,\mathrm{d}\theta}{1 + \int_0^{2\pi} f \,\mathrm{d}\theta}, \qquad f[u] = \frac{e^u}{1 + \int_0^{2\pi} f\,\mathrm{d}\theta}.$$

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$$\implies \mathrm{div}((1-\rho)\boldsymbol{p}) \in L^{2+\delta}_{t,x} \implies \partial_t\rho - \Delta\rho \in L^{2+\delta}_{t,x}$$

Calderón–Zygmund $\implies$ $\nabla\rho \in L^{2+\delta}_t W^{1,2+\delta}_x \hookrightarrow L^{2+\delta}_t L^\infty_x$ (need this for later...)

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Questions?

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